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Application of a statistical method to brittle fracture in biaxial loading systems

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The failure probability of a brittle material stressed in a tensile biaxial loading system is investigated. A "failure diagram" relating the two mean biaxial stresses is mapped out for different values of the Weibull modulus as a function of volume of material. The significance of the biaxial results in relation to the uniaxial results is discussed by defining the term "strength reduction factor" (SRF). The applicability of Weibull analysis to biaxial loading systems is also examined.

1. Introduction

Simple stress systems such as uniaxial tension, compression and bending are not frequently encountered in load bearing structures and machines. In practice, many components are subjected to biaxial stresses or combined stresses. In ductile materials, theories of predicting failure in yielding have been adequately described in many standard text books, but in a brittle material the catastrophic failure depends on the applied load, and the distribution of flaw size and orientation (with respect to applied load). Jayatilaka and Trustrum [1] analysed the failure probability for brittle materials subjected to uniaxial tensile loading. They found that the Weibull modulus, which was hitherto considered as an empirical constant, and the brittleness of a material can be related to the properties of the flaw size distribution.

In this paper the theory described by Jayatilaka and Trustrum is extended to analyse the failure in a biaxial loading system subjected to tensile stresses only. The effects of the volume of material under stress are also examined for different brittle materials, which are characterized by their Weibull moduli.

2. Theory

2.1. Strength of inclined cracks

Determination of the crack growth of a body containing an inclined crack under a uniaxial loading system has been studied by Sih [2] and Jayatilaka *et al.* [3] using strain energy density concepts. They showed that the initial crack growth is determined when the strain energy density of a body attains a minimum value. This theory was extended by Jayatilaka *et al.* to biaxial loading systems. If σ_1 and σ_2 are the biaxial stresses (see Fig. 1), β is the crack angle and ν is the Poisson's ratio, then they showed that the strength σ_2 can be expressed as

$$\sigma_2^2 a = L(\beta, R, \nu) \qquad (1)$$

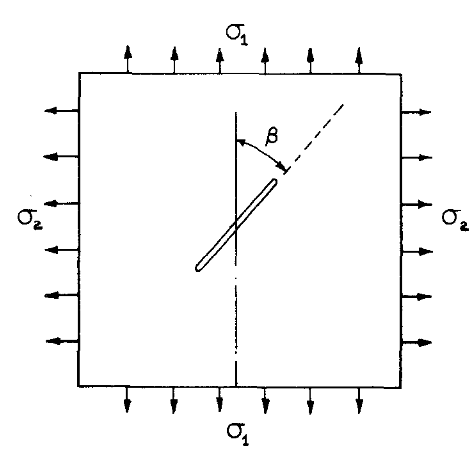

Figure 1 An inclined crack in a biaxial tensile stress system.

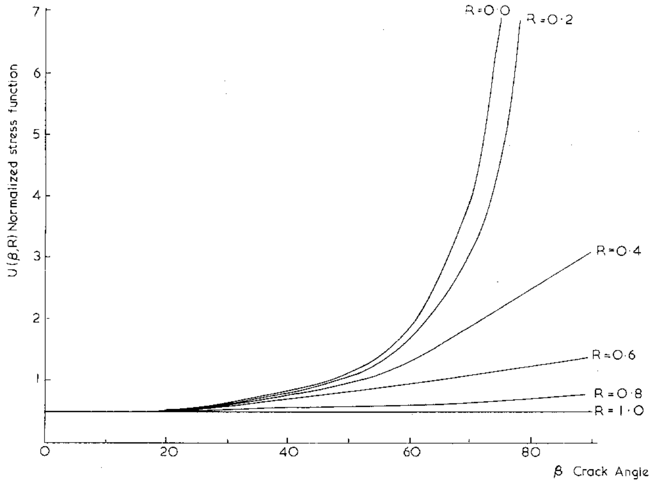

Figure 2 The family of curves given by Equation 2. The normalized stress function $U(\beta, R) = \sigma_2^2 \pi a/(2K_{IC}^2)$

where $R = \sigma_1/\sigma_2$ and a is the semi-crack length. For $\nu = 0.25$ (the value used for brittle materials), Equation 1 may be re-written [1] in terms of the critical stress intensity factor, K_{IC}, as

$$\sigma_2^2 a = \frac{2}{\pi} K_{IC}^2 U(\beta, R) \tag{2}$$

Fig. 2 shows the family of curves expressed in Equation 2 when σ_1 and σ_2 are both tensile. In the general case, when σ_1 and σ_2 are either tensile or compressive or a combination of both, a "failure diagram" (see Fig. 3) can be drawn, on the assumption that the critical flaw size in a body always lies in the direction in which σ_2 is minimum for a given value of σ_1/σ_2. A failure diagram such as this one is not a realistic estimate of the strength of a brittle material due to the random distribution of flaw sizes and their orientations within a given volume of material.

The initial crack growth direction of a body containing an inclined crack under a given loading system can be readily determined [2, 3] but the subsequent direction of crack growth cannot be determined analytically, using a known theory. It is a well known experimental fact that in the case of uniaxial tensile loading systems, the subsequent direction of crack growth, for an inclined crack, always turns away from the initial direction of propagation and runs normal to the applied load, thus producing a catastrophic failure. By contrast, in compression, the initial crack does not propagate catastrophically and grows only slowly to a

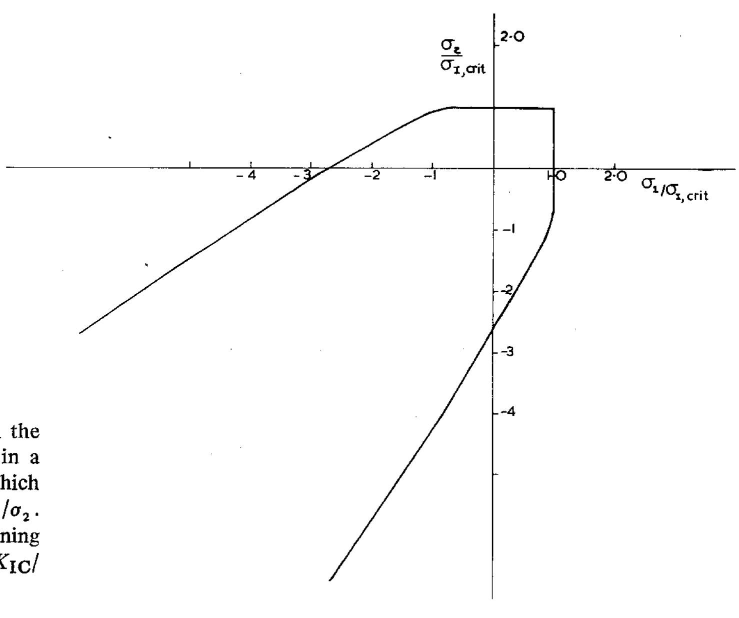

Figure 3 A "failure diagram" drawn on the assumption that the critical flaw size in a body always lies in the direction in which σ_2 is minimum for a given value of σ_1/σ_2. $\sigma_{I,crit}$ is the stress to failure for Opening Mode (uniaxial conditions) $\sigma_{I,crit} = K_{IC}/\sqrt{(\pi a_{crit})}$.

certain length, as the load is gradually increased. This experimental observation [4] can be explained since the subsequent crack path gradually aligns itself with the axis of the compression loading thus requiring a very high stress to propagate it further. It follows that in compression, failure of one flaw does not lead to total failure.

In the absence of any experimental or theoretical evidence into the nature of the fracture process for a biaxial loading system where one of the stresses is compressive, the analysis given below to determine the failure probability in biaxial loading systems is confined only to biaxial tensile loading systems ($\sigma_1 > 0$, $\sigma_2 > 0$), where failure of one flaw leads to total failure.

2.2. Statistical approach

Using Equation 1 and assuming that any crack angle β is equally likely, the probability of failure, $F(\sigma_2)$, at stress σ_2 for one crack is given by

$$F(\sigma_2) = \iint \frac{2}{\pi} f(a)\,\mathrm{d}a\,\mathrm{d}\beta$$

$$0 \leqslant \frac{L(\beta, R, \nu)}{a} \leqslant \sigma_2^2 \qquad (3)$$

$$0 \leqslant \beta \leqslant \pi/2$$

where $f(a)$ is the probability density of the semi-crack length. It was shown in [1] that $f(a)$ can be closely fitted by the expression

$$f(a) = \frac{c^{n-1}a^{-n}}{(n-2)!}\, e^{-c/a}, \text{ for } a > 0 \qquad (4)$$

where c/n is the mode of the distribution and n measures the rate at which the density tends to zero. On using Equations 2 and 4, and then substituting $V = c/a$, Equation 3 becomes

$$F(\sigma_2) = \int_0^{\pi/2}\int_0^{x} \frac{2}{\pi}\, \frac{V^{n-2}e^{-V}}{(n-2)!}\, \mathrm{d}V\mathrm{d}\beta \qquad (5)$$

where $x = \sigma_2^2 \pi c/2K_{\mathrm{IC}}^2\, U(\beta, R)$. One integration gives

$$F(\sigma_2) = \int_0^{\pi/2} (1 - e^{-x})\frac{2}{\pi}\mathrm{d}\beta, \text{ for } n = 2 \qquad (6)$$

$$F(\sigma_2) = \int_0^{\pi/2}\left[1 - e^{-x}\left(1 + x + \frac{x^2}{2}\right)\right]\frac{2}{\pi}\mathrm{d}\beta, \text{for } n = 4 \qquad (7)$$

and

$$F(\sigma_2) = \int_0^{\pi/2}\left[1 - e^{-x}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}\right)\right]\frac{2}{\pi}\mathrm{d}\beta, \quad \text{for } n = 6 \qquad (8)$$

For N cracks, the probability of failure, P_f, at stress σ_2 is given by

$$P_\mathrm{f} = 1 - [1 - F(\sigma_2)]^N, \qquad (9)$$

since $1 - P_\mathrm{f}$ is the probability of survival of all N cracks. The mean stress, $\bar{\sigma}_2$, can now be evaluated from

$$\bar{\sigma}_2 = \int_0^{\infty} (1 - P_\mathrm{f})\,\mathrm{d}\sigma_2 \qquad (10)$$

The Integrals 6, 7 and 8 were evaluated numerically for the values $R = 0$, 0.4, 0.6 and 1.0, using the computed values of $U(\beta, R)$ (see Fig. 2). Then the normalised mean stress was calculated by the numerical integration of Equation 10 for a range of values of N. Fig. 4 shows the normalized mean stress, $\bar{\sigma}_2/\sigma_\mathrm{I}$, plotted against the number of cracks for $R = 0$ and 1.0, which correspond to uniaxial and equal biaxial tensile loading, respectively. The values for $R = 0.4$ and 0.6 follow a similar pattern and lie between the $R = 0$ and 1.0 values. In Fig. 5, $\bar{\sigma}_1/\sigma_\mathrm{I}$ is plotted against $\bar{\sigma}_2/\sigma_\mathrm{I}$ for $n = 2, 4, 6$ and $N = 40, 100, 200, 400, 600$.

2.3. Relation with Weibull modulus

It follows from Expression 9 that when N is large

$$P_\mathrm{f} \simeq 1 - \exp[-NF(\sigma_2)] \qquad (11)$$

and for $x \ll 1$, Equation 5 becomes

$$F(\sigma_2) \simeq \int_0^{\pi/2} \frac{2}{\pi}\, \frac{x^{n-1}}{(n-1)!}\,\mathrm{d}\beta \qquad (12)$$

where $x = \sigma_2^2 \pi c/2K_{\mathrm{IC}}^2\, U(\beta, R)$. Thus P_f takes the form,

$$P_\mathrm{f} \simeq 1 - \exp[-Nk_1(n, R)\,\sigma_2^{2n-2}] \qquad (13)$$

where

$$k_1(n, R) = \int_0^{\pi/2}\left[\frac{\pi c}{2K_{\mathrm{IC}}^2\, U(\beta, R)}\right]^{n-1} \frac{2}{\pi(n-1)!}\mathrm{d}\beta \qquad (14)$$

The approximate expression (13) for P_f is identical to that used in Weibull analysis, when the threshold stress is assumed to be zero, as for most brittle materials. Consequently the relation

$$m = 2n - 2 \qquad (15)$$

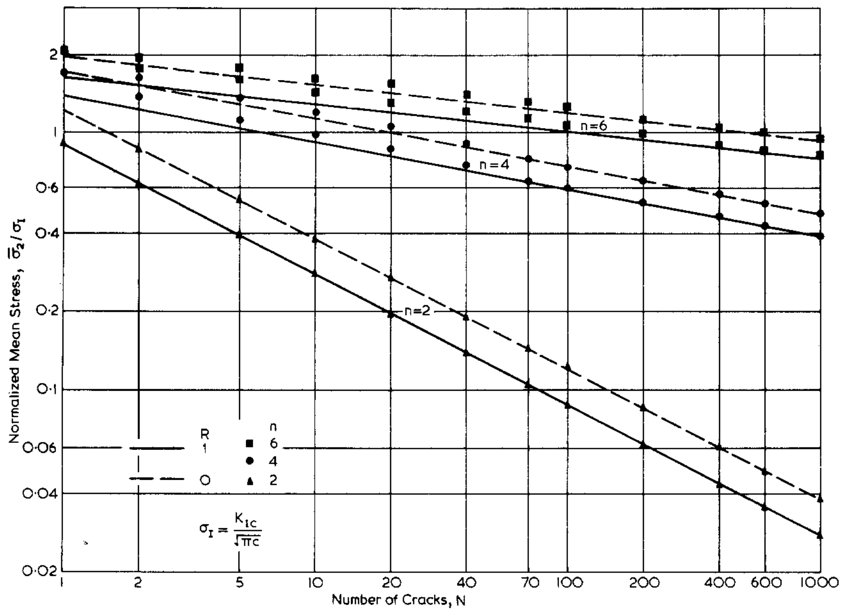

Figure 4 Logarithmic plot of normalized mean stress in terms of number of cracks. Straight lines correspond to the "Weibull slopes".

between the Weibull modulus m and the parameter n of the crack size distribution found by [1] can be extended to biaxial loading. In the case $n = 2$ and $R = 1$ for which $U(\beta, R) = 0.5$ (see Fig. 2), it follows from Equations 6 and 9 that

$$F(\sigma_2) = 1 - \exp(-\sigma_2^2 \pi c/K_{IC}^2) \quad (16)$$

and

$$P_f = 1 - \exp(-N\sigma_2^2 \pi c/K_{IC}^2) \quad (17)$$

so the strength, σ_2, exactly follows the Weibull distribution with $m = 2$.

The validity of this approximation can be tested by plotting $\bar{\sigma}_2$ against N, since for the Weibull distribution (13),

$$\bar{\sigma}_2 = [Nk_1(n, R)]^{-1/(2n-2)}\, \Gamma\left(1 + \frac{1}{2n-2}\right) \quad (18)$$

or

$$\log \bar{\sigma}_2 = \log \Gamma\left(1 + \frac{1}{2n-2}\right) - \frac{1}{2n-2}\log k_1(n, R) - \frac{1}{2n-2}\log N \quad (19)$$

where Γ is the "gamma" function. Equation 19 shows that for a good approximation the graph of $\log \bar{\sigma}_2$ against $\log N$, should be a straight line of gradient $(2n - 2)^{-1}$, whose intercept varies with n and R. In Fig. 4 straight lines of slopes 1/2, 1/6, 1/10 corresponding to $n = 2$, 4, and 6, have been drawn for comparison. The results show that the approximation is excellent for $n = 2$ and reasonable for $n = 4$ and 6, provided $N > 100$.

As was noted in [1], even if c is replaced by the mode, c/n, of the distribution of crack size in the definition of σ_I, the mean strength, $\bar{\sigma}_2$, for the smaller values of n is less and hence materials with lower values of the Weibull modulus m are more brittle. This can be explained in terms of the greater variability of crack size for small n and the consequent greater probability of a large crack being present.

3. Discussion

Previous statistical methods to determine the strength of a brittle material, based on empirical data, are limited to uniaxial tensile loading systems. Hence, an important outcome of the work described in this paper is the development of a theory that cannot only be applied to uniaxial tensile loading systems, but also to more complex, biaxial tensile loading conditions.

The effect of a biaxial stress on the fracture strength of a brittle body, containing randomly orientated flaws, can be observed from Fig. 5. On further analysis, this effect can be best explained with respect to the uniaxial stress effects by defining a suitable term, called the "strength

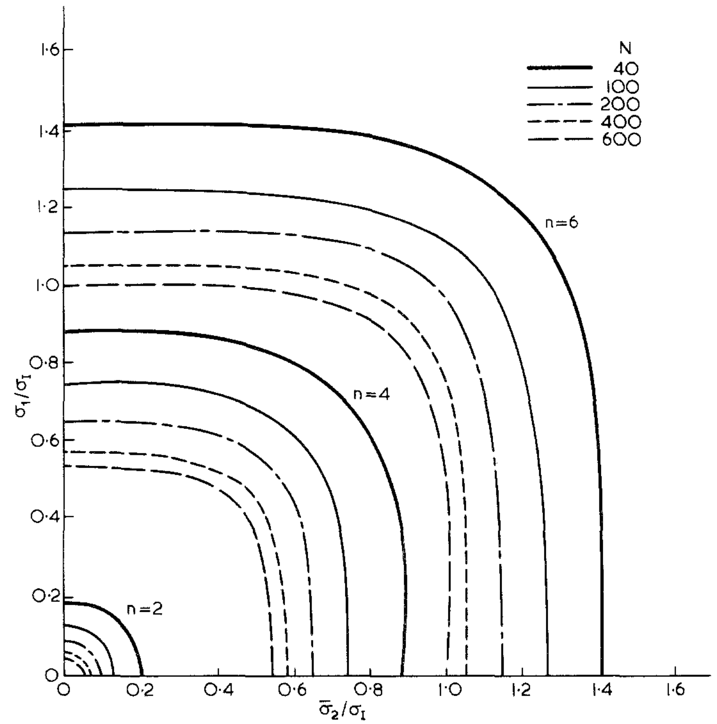

Figure 5 A "failure diagram" relating the two normalized mean biaxial stresses.

reduction factor" (SRF), as equal to the ratio of the mean failure stresses (higher stress in the case of biaxial loading) in biaxial loading to uniaxial loading conditions. That is,

$$\text{SRF} = \frac{\text{mean failure stress in biaxial loading}}{\text{mean failure stress in uniaxial loading.}}$$

Table I shows a set of values for SRF under equal biaxial stresses ($R = 1$) as a function of number of cracks. It is evident that for a given material the SRF remains fairly constant for the range of cracks considered. Thus, it leads to an important result that it is possible to postulate values for SRF (see Table II), independent of the volume, for different materials. These values should prove to be very useful to a design engineer who, very often, has to design structures subjected to biaxial stresses using a knowledge of failure characteristics under uniaxial conditions; the failure stresses in uniaxial conditions can be readily obtained from handbooks or by performing simple experiments.

The result that the SRF is almost independent of volume also gives support to the validity of the approximations used to derive the Weibull distribution (13).

Using Equation 18,

$$\text{SRF} = \frac{\bar{\sigma}_2(R)}{\bar{\sigma}_2(0)} = \left[\frac{k_1(n, R)}{k_1(n, 0)}\right]^{-1/(2n-2)} \quad (20)$$

which shows that the SRF is only a function of n and R and so independent of the number of cracks, N, and hence of the volume. In particular when $R = 1$, an explicit expression can be found for the SRF. It follows from Equation 14 that

TABLE I Values of SRF for equal biaxial loading ($R = 1$) conditions.

N	SRF (%)		
	$n = 2$ ($m = 2$)	$n = 4$ ($m = 6$)	$n = 6$ ($m = 10$)
40	71.7	81.4	84.6
70	72.0	81.6	85.2
100	72.1	81.9	85.3
200	72.0	82.4	85.8
400	72.0	82.5	86.1
600	72.1	82.6	86.3
1000	72.0	82.6	86.6

TABLE II Average values of SRF for different biaxial loading systems. The values given in parentheses refer to the standard deviations.

n	m	SFR (%)		
		$R = 1.0$	$R = 0.6$	$R = 0.4$
2	2	71.7 (0.1)	86.9 (0.1)	93.6 (0.4)
4	6	82.1 (0.5)	95.5 (0.2)	98.6 (0.1)
6	10	85.7 (0.7)	97.3 (0.4)	99.3 (0.2)

$$\frac{k_1(n,1)}{k_1(n,0)} = \frac{\int_0^{\pi/2} [U(\beta,1)]^{1-n} \mathrm{d}\beta}{\int_0^{\pi/2} [U(\beta,0)]^{1-n} \mathrm{d}\beta} \tag{21}$$

Since $U(\beta, 1) = 0.5$ (see Fig. 2) and $U(\beta, 0) \simeq \frac{\pi}{4}\left(\frac{\pi}{2} - \beta\right)^{-1}$ (see [1]),

$$\frac{k_1(n,1)}{k_1(n,0)} = \frac{\pi 2^{n-2}}{\pi n^{-1} 2^{n-2}} = n \tag{22}$$

and hence,

$$\mathrm{SRF} = n^{-1/(2n-2)} \tag{23}$$

The values for the SRF given by Equation 23 for $n = 2, 4, 6$ are 70.7%, 79.4% and 83.6%, respectively, which are close to the values given in Table I.

The Expression 23 shows that the strength reduction factor apporaches unity as n increases, i.e. for large values of the Weibull modulus. This can be explained by observing that the variability of crack size in the assumed model, given by Equation 4, is smaller for the larger values of n. Consequently, in any direction the most critical crack is of roughly the same size and hence the fracture of a brittle body is only slightly reduced by making the stress system biaxial rather than uniaxial.

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